Quantum integrability of nonultralocal models through Baxterisation of quantised braided algebra

LADISLAV HLAVATÝ
Faculty of Nuclear Sciences and Physical Engineering, Prague *

and

ANJAN KUNDU†
Physikalisches Institut der Universität Bonn, 53115 Bonn, Germany

Abstract

A scheme suitable for describing quantum nonultralocal models including supersymmetric ones is proposed. Braided algebras are generalised to be used through Baxterisation for constructing braided quantum Yang–Baxter equations. Supersymmetric and some known nonultralocal models are derived in the framework of the present approach. As further applications of this scheme construction of new quantum integrable nonultralocal models like mKdV and anyonic supersymmetric models including deformed anyonic super algebra are outlined.

*Postal address: Brežová 7, 115 19 Prague 1, Czech Republic. e-mail: hlavaty@br.fjfi.cvut.cz
†Permanent address: Theory Division, Saha Institute of Nuclear Physics, AF/1 Bidhan Nagar, 700 064 Calcutta, India. e-mail: Kundu@saha.ernet.in
1 Introduction

One of the major achievements in the theory of integrable systems is its extension from the classical to the quantum domain [1] and exact solution of a number of quantum models [2]. However, inspite of this success the basic theory has been developed only for a limited class of quantum models, known as ultralocal models [1]. For such systems the Lax operators at different lattice points $j, k; j \neq k$ must commute, i.e.

$$L_{2k}(v)L_{1j}(u) = L_{1j}(u)L_{2k}(v)$$  \hspace{1cm} (1.1)

and only under such constraint the quantum Yang-Baxter equation (QYBE)

$$R_{12}(u - v)L_{1j}(u)L_{2j}(v) = L_{2j}(v)L_{1j}(u)R_{12}(u - v), \quad j = 3, \ldots, N$$  \hspace{1cm} (1.2)

can be raised to the level of monodromy matrix

$$T_{a}(u) = L_{aN}(u)L_{a,k-1}(u) \ldots L_{a3}(u), \quad a = 1, 2$$  \hspace{1cm} (1.3)

yielding the corresponding QYBE

$$R_{12}(u - v)T_{1}(u)T_{2}(v) = T_{2}(v)T_{1}(u)R_{12}(u - v)$$  \hspace{1cm} (1.4)

The trace of equation (1.4) yields a set of commuting operators $[tr_{1}T_{1}(u), tr_{2}T_{2}(v)] = 0$, ensuring the exact integrability of the corresponding quantum system.

On the other hand there exists a rich class of nonultralocal models [3] including mKdV, KdV, DNLS, chiral models etc., which are classically integrable, though for them no general quantum theory is available up to this date. In the last ten years several nonultralocal models were presented at the quantum level in different contexts, extending from integrable systems [4, 5] to conformal field theory (CFT) related models [6, 7, 8, 9]. However, as far as we know, there was no major effort to combine them as a theory, except only some ingenuous proposals for quantum maps [5] and ’ultralocalisable’ nonultralocal theories [10]. It is therefore highly desirable to formulate a quantum theory of nonultralocal models, at least for a certain class, which would permit, in parallel to the ultralocal case, their systematic construction starting from some fundamental level. Such a theory should include naturally the supersymmetric models, since they are also nonultralocal in a broader sense. For achieving this nontrivial task however, one requires clearly to generalise suitably the basic structures like quantum algebra [11], Faddev-Reshetikhin-Takhtajan (FRT) algebra [12], Yang-Baxter equation (YBE) etc. With this global aim in mind we propose here some primary steps in this direction.

The arrangement of the paper is as follows. In sect. 2 we generalise the notion of quantised braided group [13] and its dual algebra leading to a braided extension of the FRT algebra. This FRT algebra is found to permit Baxterisation much in common with the ultralocal case [14], yielding a spectral parameter dependent braided QYBE for the Lax operator of the nonultralocal lattice models. This is the content of sect. 3. In sect. 4 we derive the global QYBE for the monodromy matrix of the corresponding
periodic models using the associated bialgebra structure. The quantum integrability is
given by the commuting set of operators. Their construction from the QYBE, which
becomes nontrivial here, is solved in sect. 5 for a certain class of braidings. In sect.
6 we find that the quantum supersymmetric and nonultralocal models, like nonabelian
Toda chain [4], lattice Gelfand-Dikii mapping [5] as well as CFT related models like
fit well in our proposed theory. To illustrate further the usefulness of this scheme we
construct in sect. 7 new examples of quantum integrable ultralocal models such as
quantum lattice generalisation of mKdV model, a new type of supersymmetric model
involving bosons and anyons and a quantum deformation of the anyonic super algebra.
Sect. 8 is the concluding section.

2 Generalised quantised braided group and its
dual algebra

The quantised braided groups were introduced in [13] combining Majid’s concept of
braided groups [15] and the FRT formulation of quantum supergroups [16]. The genera-
tors of quantised braided groups $T = T_i^j$, $i, j \in \{1, \ldots, d = \dim V\}$ satisfy the
relation

$$R_{12}Z_{12}^{-1}T_1Z_{12}T_2 = Z_{21}^{-1}T_2Z_{21}T_1R_{12}$$

(2.1)

where the numerical matrices satisfy the system of Yang–Baxter–type equations

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

(2.2)

$$Z_{12}Z_{13}Z_{23} = Z_{23}Z_{13}Z_{12}.$$  

(2.3)

$$R_{12}Z_{13}Z_{23} = Z_{23}Z_{13}R_{12}, \quad Z_{12}Z_{13}R_{23} = R_{23}Z_{13}Z_{12}$$

(2.4)

The special cases are the quantum supergroups [16], where $Z_{ij}^{kl} = (-)^{i\dot{j}}\delta_i^k\delta_j^l$ and braided
linear groups [17], where $R = Z$ and $\dot{i} = \text{grad}(i)$.

The dual algebra is generated by the generators $L^\pm = \{L_i^\pm \}$ that satisfy

$$R_{21}Z_{21}^{-1}L_1^\epsilon Z_{21}L_2^\sigma = Z_{12}^{-1}L_2^\sigma Z_{12}L_1^\epsilon R_{21}$$

(2.5)

where $(\epsilon, \sigma) = (+, +), (+, -), (-, +), (-, -)$. The fundamental representation $\phi_3$ of this algebra
in $V_3$ such that $\dim V_3 = \dim V_2 = \dim V_1$ is

$$\phi_3(L_a^+) = Z_{a3}R_{3a}, \quad \phi_3(L_a^-) = Z_{a3}R_{3a}^{-1} \quad a = 1, 2.$$  

(2.6)

The algebras (2.1) and (2.5) can be turned into bialgebras by introducing matrix
coproducts

$$\Delta(T_i^j) = T_i^k \otimes T_k^j, \quad \Delta(L_i^\epsilon) = L_i^\epsilon \otimes L_i^\epsilon$$

(2.7)

provided the multiplication in the tensor product of algebras is defined by virtue of a
braid map given by matrix $Z$ as

$$T_2^{(k)}Z_{12}^{-1}T_1^{(j)} = Z_{12}^{-1}T_1^{(j)}Z_{12}T_2^{(k)}Z_{12}^{-1}$$

(2.8)
\[ L^\epsilon_1(j) Z_{21}^{-1} L^\epsilon_2(j) = Z_{21}^{-1} L^\epsilon_1(j) Z_{21} L^\epsilon_2(j) Z_{21}^{-1} \] (2.9)

for \( j < k \) and \( \epsilon_1, \epsilon_2 \) being all possible combinations of \((+, -)\). Here \( L^{(j)}, T^{(j)} \) are copies of the generators \( L^j, T \) in the \( j \)-th factor of the tensor product of algebras.

Now to make the theory suitable for application to nonultralocal quantum systems we need algebras with braiding maps acting differently on different pairs of factors in the tensor product. For this purpose we propose a generalisation of the quantised braided group in the following way.

\[ R_{12}Z_{12}^{-1}T_1\tilde{Z}_{12}T_2 = Z_{21}^{-1}T_2\tilde{Z}_{21}T_1R_{12} \] (2.10)

where \( Z \) can be different from \( \tilde{Z} \).

The algebra dual to (2.10) is generated by \( L^\pm = \{ L_i^\pm j \} \) and satisfy

\[ R_{21}Z_{21}^{-1} L_i^\epsilon \tilde{Z}_{21} L_2^\sigma = Z_{12}^{-1} L_2^\sigma \tilde{Z}_{12} L_2^\epsilon R_{21} \] (2.11)

where \( (\epsilon, \sigma) = (+, +), (+, -), (-, -) \), which is a braided generalisation of the FRT algebra [12]. The possibility of introducing coproduct into the generalised FRT algebra (2.11) requires braiding relations for the generators \( L^\pm \) as

\[ L^\epsilon_{1(j+1)} Z_{21}^{-1} L^\epsilon_1 = Z_{21}^{-1} L^\epsilon_1 Z_{21} L^\epsilon_{1(j+1)} \tilde{Z}_{21}^{-1} \] (2.12)

and

\[ L^\epsilon_{1(k)} \tilde{Z}_{21}^{-1} L^\epsilon_1 = \tilde{Z}_{21}^{-1} L^\epsilon_1 \tilde{Z}_{21} L^\epsilon_{1(k)} \tilde{Z}_{21}^{-1} \] (2.13)

for \( k > j + 1 \). Note that the relations (2.12),(2.13) are generalisation of (2.9) that distinguishes between nearest and non-nearest neighbours.

The numerical matrices \( R_{12}, \tilde{Z}_{12}, Z_{12} \) satisfy a system of Yang-Baxter type equations generalising (2.2)-(2.4) in the form

\[ R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \] (2.14)
\[ Z_{12}Z_{13}Z_{23} = Z_{23}Z_{13}Z_{12}, \] (2.15)
\[ \tilde{Z}_{12}\tilde{Z}_{13}\tilde{Z}_{23} = \tilde{Z}_{23}\tilde{Z}_{13}\tilde{Z}_{12} \] (2.16)
\[ R_{12}\tilde{Z}_{13}\tilde{Z}_{23} = \tilde{Z}_{23}\tilde{Z}_{13}R_{12}, \quad \tilde{Z}_{12}\tilde{Z}_{13}R_{23} = R_{23}\tilde{Z}_{13}\tilde{Z}_{12} \] (2.17)
\[ Z_{12}\tilde{Z}_{13}\tilde{Z}_{23} = \tilde{Z}_{23}\tilde{Z}_{13}Z_{12}, \quad \tilde{Z}_{12}\tilde{Z}_{13}Z_{23} = Z_{23}\tilde{Z}_{13}\tilde{Z}_{12} \] (2.18)
\[ R_{12}Z_{13}Z_{23} = Z_{23}Z_{13}R_{12}, \quad Z_{12}Z_{13}R_{23} = R_{23}Z_{13}Z_{12} \] (2.19)

These equations guarantee the associativity of the triple product

\[ L^\epsilon_1(j)(\tilde{Z}_{21} L^\epsilon_2(j))\tilde{Z}_{31} L^\epsilon_3(j)) \] (2.20)

Actually the set of equations following directly from the associativity condition is a bit different, but they are satisfied when the system (2.14)-(2.19) holds. For example in the alternative set of equations (2.15) does not appear, but on the other hand one gets

\[ \tilde{R}_{12}R_{13}R_{23} = R_{23}\tilde{R}_{13}R_{12}, \] (2.21)

where \( \tilde{R}_{12} = Z_{12}R_{12}Z_{21}^{-1} \).
3 Baxterisation of braided FRT algebra

For framing our theory applicable to integrable systems we need to introduce spectral parameter into the above algebraic structures. That can be done by Baxterisation procedure [14].

Introducing $R_{12}^+ = R_{21}$ and $R^- = P(R^+)^{-1}P$ with $P$ being the permutation matrix, we rewrite the braided FRT algebra (2.11) in the following form, convenient for the Baxterisation.

$$R_{12}^+ Z_{21}^{-1} L_1^e \tilde{Z}_{21} L_2^e = Z_{12}^{-1} L_2^e \tilde{Z}_{12} L_1^e R_{12}^+$$

and

$$R_{12}^- Z_{21}^{-1} L_1^e \tilde{Z}_{21} L_2^e = Z_{12}^{-1} L_2^e \tilde{Z}_{12} L_1^e R_{12}^-$$

where $(\epsilon, \sigma) = (+, +), (+, -), (-, -)$.

Now to build spectral parameter dependent Lax operator and the quantum $R$-matrix satisfying the corresponding QYBE we set [14]

$$R_{12}(u) = e^u R_{12}^+ - e^{-u} R_{12}^-$$

and

$$L_a(u) = e^u L_a^+ - e^{-u} L_a^-,$$  \hspace{1cm} a = 1, 2

and observe that the Baxterisation goes through, only when apart from (3.1) and (3.2) the following extra relation

$$R_{12}^- Z_{21}^{-1} L_1^e \tilde{Z}_{21} L_2^- - R_{12}^+ Z_{21}^{-1} L_1^e \tilde{Z}_{21} L_2^+ = Z_{12}^{-1} L_2^e \tilde{Z}_{12} L_1^e R_{12}^- - Z_{12}^{-1} L_2^e \tilde{Z}_{12} L_1^e R_{12}^+$$

holds. However in analogy with [14] one finds that the above relation is satisfied under the Hecke condition

$$R^+ - R^- = cP$$

on $R^\pm$ matrices, which leads finally to the spectral parameter dependent braided equation

$$R_{12}(u - v) Z_{21}^{-1} L_1(u) \tilde{Z}_{21} L_2(v) = Z_{12}^{-1} L_2(v) \tilde{Z}_{12} L_1(u) R_{12}(u - v)$$

with $R_{12}(u)$ satisfying the standard YBE

$$R_{12}(u - v) R_{13}(u) R_{23}(v) = R_{23}(v) R_{13}(u) R_{12}(u - v)$$

along with the equations

$$R_{21}(u) Z_{13} Z_{23} = Z_{23} Z_{13} R_{21}(u)$$

$$Z_{12} Z_{13} R_{32}(u) = R_{32}(u) Z_{13} Z_{12}$$

and similar equations for $\tilde{Z}_{12}$. We would like to mention here that the Baxterised solutions (3.3) of YBE satisfy automatically the unitarity condition

$$R(u) P R(-u) P = (c^2 - 4 \sinh^2 u) 1.$$
4 Braided QYBE’s

For application of the above formulated generalised braided algebras to integrable systems we define the Lax operators $L_j(u)$ of the associated discrete model which satisfy (3.7) as the braided local QYBE

$$R_{12}(u - v)Z_{21}^{-1}L_{1j}(u)\tilde{Z}_{21}L_{2j}(v) = Z_{12}^{-1}L_{2j}(v)\tilde{Z}_{12}L_{1j}(u)R_{12}(u - v). \tag{4.1}$$

For the construction of models we need a monodromy matrix $T_a^{[k,j]}(u), \ k > j$ defined as a product of the Lax operators

$$T_a^{[k,j]}(u) = L_{ak}(u)L_{a,k-1}(u) \ldots L_{aj}(u) \tag{4.2}$$

acting in the space $\mathcal{H} \equiv V_k \otimes V_{k-1} \otimes \ldots \otimes V_j$ and satisfying the same relation as the $L_{aj}(u)$ operators. This can be achieved if the Lax operators along with (4.1), satisfy also the Baxterised forms of braiding relations (2.12), (2.13):

$$L_{2j+1}(v)Z_{21}^{-1}L_{1j}(u) = \tilde{Z}_{21}^{-1}L_{1j}(u)\tilde{Z}_{21}L_{2j+1}(v)\tilde{Z}_{21}^{-1} \tag{4.3}$$

and

$$L_{2k}(v)\tilde{Z}_{21}^{-1}L_{1j}(u) = \tilde{Z}_{21}^{-1}L_{1j}(u)\tilde{Z}_{21}L_{2k}(v)\tilde{Z}_{21}^{-1} \tag{4.4}$$

for $k > j + 1$.

Using now the relations (4.1), (4.3) and (4.4) it is straightforward to show that the monodromy matrix satisfies the equation

$$R_{12}(u - v)Z_{21}^{-1}T_1^{[k,j]}(u)\tilde{Z}_{21}T_2^{[k,j]}(v) = Z_{12}^{-1}T_2^{[k,j]}(v)\tilde{Z}_{12}T_1^{[k,j]}(u)R_{12}(u - v) \tag{4.5}$$

Note that the ultralocal condition (1.1) is generalised here to the braiding relations (4.3) and (4.4), which carry the main feature of nonultralocality. It can be seen from (4.3), (4.4) that the present theory is designed to cover models with nonultralocal commutation relations different for nearest and nonnearest neighbours.

We intend to incorporate now the periodic boundary condition $L_{aj+p}(u) = L_{aj}(u)$ to approach closer to the physical models and consider the global monodromy matrix $T_a(u) \equiv T_a^{[N,3]}(u)$ for the closed chain $[N,3]$. Observe however that for deriving equation for the monodromy matrix $T_a$ one requires along with (4.3) and (4.4) also the braiding relation like

$$L_{2j-1}(v)\tilde{Z}_{21}^{-1}L_{1j}(u) = \tilde{Z}_{21}^{-1}L_{1j}(u)Z_{12}^{-1}L_{2j-1}(v)\tilde{Z}_{21}^{-1} \tag{4.6}$$

the compatibility of which with (4.3) demands the constraint

$$\tilde{Z}_{12}\tilde{Z}_{21} = 1. \tag{4.7}$$

Since $L_{3-1}(u) = L_N(u)$, using (4.6) for $j = 3$ together with (4.3), (4.4) we obtain finally the braided QYBE for the monodromy matrix as

$$R_{12}(u - v)Z_{21}^{-1}T_1(u)\tilde{Z}_{21}^{-1}T_2(v) = Z_{12}^{-1}T_2(v)\tilde{Z}_{12}^{-1}T_1(u)R_{12}(u - v) \tag{4.8}$$
5 Trace identity and quantum integrability

For ensuring quantum integrability we need a commuting set of conserved quantities. By taking the trace of (4.8) we arrive at

$$tr_{12} \left( Z_{21}^{-1} T_1(u) Z_{12}^{-1} T_2(v) \right) = tr_{12} \left( Z_{12}^{-1} T_2(v) Z_{21}^{-1} T_1(u) \right)$$  \hspace{1cm} (5.1)

For $Z = 1$, i.e. for unbraided or equivalently ultralocal case, the above trace identity is trivially factorised yielding the required commuting set of operators. However for general $Z$ the trace factorisation problem, namely reducing the trace identity (5.1) to the form

$$tr_1(T_1(u))tr_2(T_2(v)) = tr_2(T_2(v))tr_1(T_1(u))$$

becomes nontrivial. For solving this problem we propose the following factorisation procedures.

5.1 Factorisation by k–trace

This procedure is based on the generalisation of Sklyanin’s approach to reflection-type algebras [18]. It was shown in [19] that if $T_i^j(u), K_i^j(u)$ are generators of the algebra satisfying quadratic relations

$$T_1(u)K_2(v) = K_2(v)T_1(u)$$  \hspace{1cm} (5.2)

$$A_{12}(u, v)T_1(u)B_{12}(u, v)T_2(v) = T_2(v)C_{12}(u, v)T_1(u)D_{12}(u, v),$$  \hspace{1cm} (5.3)

$$A_{12}^{-T}(u, v)K_1^{t_1}(u)B_{12}(u, v)K_2^{t_2}(v) = K_2^{t_2}(v)C_{12}(u, v)K_1^{t_1}(u)D_{12}^{-T}(u, v)$$  \hspace{1cm} (5.4)

where

$$B_{12}(u, v) = ((B_{12}^{t_1}(u, v))^{-1})^{t_2}, \quad C_{12}(u, v) = ((C_{12}^{t_2}(u, v))^{-1})^{t_1}$$  \hspace{1cm} (5.5)

with $A, B, C, D$ numerical matrix functions, then the 'k–trace' $t(u) = K_i^j(u)T_i^j(u) = tr[K(u)T(u)]$ forms a commutative subalgebra

$$[t(u), t(v)] = 0.$$  

In our case of (4.8)

$$A(u, v) = ZR(u - v)PZ^{-1}P, \quad B = Z^{-1}, \quad C = PZ^{-1}P, \quad D(u, v) = R(u - v).$$  

Therefore we have to find a set of operators $K_i^j(u)$ on $\mathcal{H}$ commuting with $T_i^j(u)$ and satisfying (5.4).

The simplest possibility is $K_i^j(u) = k_i^j(u)1_{\mathcal{H}}$ where $k_i^j(u)$ are elements of the numerical matrix $k(u)$ satisfying

$$k_1(u)Z_{21}k_2(v)Z_{12}R_{12}(u - v) = R_{12}(u - v)k_2(v)Z_{12}k_1(u)Z_{21}.$$  \hspace{1cm} (5.6)

where

$$\hat{Z} = (((Z^{-1})^{t_1})^{-1})^{t_1}.$$  \hspace{1cm} (5.7)
It is true that equation (5.6) determining the condition for factorisability of traces is rather difficult to solve in the general case. Nevertheless, in our attempt to simplify this equation we observe interestingly that, for some special ansätze of \( Z \) one may obtain \( \mathcal{Z} = Z \). These ansätze for \( Z \) are

a) the diagonal form

b) factorised form: \( Z = A \otimes B \)

c) specifically chosen form:

\[
Z_{12} = 1 + \sum_i A_i \otimes B_i, \quad (B_i)^2 = 0,
\]  

(5.8)

d) \( Z \) coinciding with \( R_0 \):

\[
Z_{12} = R^{(2)}_{q21},
\]  

(5.9)

e) \( Z \) expressed as

\[
Z_{12} = e^h \sum_i A_i \otimes B_i.
\]  

(5.10)

with arbitrary invertible matrices \( A_i, B_i \). The related reduction \( \mathcal{Z} = Z \) allows to find now some particular solutions of the factorisablity equation (5.6) as \( k = 1 \) under the condition

\[
R_{12}(u)Z_{12}Z_{21} = Z_{21}Z_{12}R_{12}(u)
\]  

(5.11)

causing the ordinary trace of \( T(u) \) to produce the set of commuting operators.

Let us now have a closer look into particular cases of the above proposal, while another possibility of factorisation will be given below. Finally we will show that the known nonultralocal as well as supersymmetric models fit well into these solutions.

1. Let \( Z \) be a diagonal matrix (case a) above):

\[
Z_{12} = \sum_{\alpha, \beta} g_{\alpha \beta} e_{\alpha \alpha} \otimes e_{\beta \beta}, \quad \text{where} \quad (e_{\alpha \beta})_{\gamma \delta} = \delta_{\alpha \gamma} \delta_{\beta \delta}.
\]  

(5.12)

If

\[
g_{\alpha \beta} g_{\beta \alpha} = 1,
\]  

(5.13)

then (5.11) is trivially satisfied giving \( k = 1 \), i.e. commutativity of \( trT \). We show below that the condition (5.13) holds in the supersymmetric case and one can construct SUSY invariant integrable models with the solution (5.12) and \( \mathcal{Z} = Z \). On the other hand when \( g_{\alpha \beta} = g_{\beta \alpha} \) is symmetric, it commutes with the \( R \)-matrix of the class (6.1) and hence leads to the 'k-trace' with arbitrary \( k \) commuting with such \( R \) and \( Z \)-matrices. Such a possibility will be applied in sect. 7.2–3 for constructing anyonic quantum integrable models.

2. Let \( Z_{12} \) be in the factorised form \( Z_{12}^{-1} = A_1 B_2 \) (case b) above).

The equation (5.6) in this case can be solved easily for arbitrary \( R \) yielding \( k = AB \). Therefore as follows from the above result, the 'k-trace' or \( tr\tilde{T} = tr(ABT) \) would generate the commuting conserved quantities. This conclusion can also be confirmed by inserting the factorised form of \( Z \) directly in our trace formula (5.1).
3. Let $Z_{12}$ exhibits the symmetry (5.11) and restricted as in the case c) above.
We see that in this case $k = 1$ and using (5.11) the braided QYBE's (4.1), (4.8) may be transformed to

$$R_{12}(u - v)Z_{12}L_{1j}(u)Z_{2j}(v) = Z_{21}L_{2j}(v)Z_{12}L_{1j}(u)R_{12}(u - v)$$  (5.14)

and

$$R_{12}(u - v)Z_{12}T_{1}(u)Z_{12}^{-1}T_{2}(v) = Z_{21}T_{2}(v)Z_{21}^{-1}T_{1}(u)R_{12}(u - v).$$  (5.15)

Due to $k = 1$, $trT$ will now exhibit commuting relations, which we have also checked independently by taking trace of (5.15) and using the form (5.8). A concrete realisation of this case will be shown below through an example.

4. Let $Z_{12} = R_{q21}^{+}$ (case d) above
Using the Baxterisation formula (3.3) and the relation $R_{12}^{-1} = (R_{21}^{+})^{-1}$ we may check that this is a solution of (5.11) for $Z$ so that $k = 1$, which ensures the trace factorisability. Moreover the same choice for $Z$ satisfies also the eqns. (2.15) and (2.19) for $R_{12} = R_{q21}^{+}$ and consequently, it can serve as a right candidate for constructing nonultralocal quantum integrable models.

To see how the relations are converted in this special case, which will be considered in the next section, we use the relation $R_{q12}^{+} = (R_{q12}^{+})^{-1} = R_{q21}^{-1} = Z_{12}^{+}$ with $t = q^{-1}$ and rewrite (3.1) as

$$(R_{q12}^{+})^{-1}R_{q12}^{+}L_{ij}^{\xi}Z_{21}L_{2j}^{\sigma} = (R_{q12}^{-1})^{-1}L_{2j}^{\sigma}Z_{12}L_{ij}^{\xi}(R_{q12}^{+})^{-1}$$

or

$$R_{q12}^{-1}L_{ij}^{\xi}Z_{21}L_{2j}^{\sigma}R_{q12}^{+} = L_{2j}^{\sigma}Z_{12}L_{ij}^{\xi}$$  (5.16)

and the braiding relations (2.12-2.13) as

$$L_{2j+1}^{\xi_{1}}R_{q12}^{+}L_{1j}^{\xi_{2}} = Z_{21}^{-1}L_{1j}^{\xi_{2}}Z_{21}L_{2j+1}^{\xi_{1}}Z_{21}^{-1}, \quad L_{2k}^{\xi_{1}}Z_{21}^{-1}L_{1j}^{\xi_{2}} = Z_{21}^{-1}L_{1j}^{\xi_{2}}Z_{21}L_{2k}^{\xi_{1}}Z_{21}^{-1}$$  (5.17)

for $k > j + 1$.

The expression for $\tilde{Z} \neq 1$ must be obtained by solving its relevant equations. For example, the simplest choice in this case may be $\tilde{Z} = 1$ or $\tilde{Z}_{12} = Z_{12} = R_{21}^{+}$.

5. Let $Z_{12} = e^{-\frac{1}{2}h \sum_{i} H_{i} \otimes H_{i}}$, where $H_{i}$'s are commuting matrices (case e) above with $A = B$)
In this case $Z_{12} = Z_{21} = \tilde{Z}_{12}$ and for $k_{i}$ commuting with $Z_{12}$ the equation (5.6) reduces to

$$[R(u - v), C(u, v)] = 0, \quad C(u, v) = Z_{12}^{2}k_{1}(u)k_{2}(v).$$  (5.18)

If now we limit ourselves to the trigonometric $R(u - v)$-matrix (6.1) and consider

$H_{i} = e_{ii} - e_{i+i+1+1}$, then we may extract a solution $k = e^{-\frac{1}{2}h \sum_{i} H_{i}^{2}}$, since $C_{12} = Z_{12}^{2}k_{1}k_{2} = e^{-\frac{1}{2}h \sum_{i} (H_{i} + H_{i}^{2})^{2}}$ and $H_{i} + H_{i+1}$ commutes with such $R(u - v)$-matrix.
It is worth noting that defining $\tilde{L}_{ij} = k^{-\frac{1}{2}}L_{ij}$ we can eliminate in this case the $Z$ dependence from the braided QYBE (4.1) yielding

$$R_{12}(u - v)\tilde{L}_{1j}(u)\tilde{Z}_{2l}(v) = \tilde{L}_{2j}(v)\tilde{Z}_{1l}(u)R_{12}(u - v)$$  \hspace{2cm} (5.19)

and similar simplifications can be achieved in (4.5),(4.8).

We will see below how some known nonultralocal models are realised as cases 4 and 5. There are however other models that go beyond the above ansätze, for which we present here another factorisation procedure.

## 5.2 Factorisation by 'gauge transformation'

Let $\hat{l}$ is a matrix of operators acting in the Hilbert space satisfying the relation

$$[R_{12}(u), \hat{l}_1\hat{l}_2] = 0 \quad [\hat{l}_1, \hat{l}_2] = 0$$  \hspace{2cm} (5.20)

and

$$Z_{21}^{-1}T_1 = \hat{l}_2T_1\hat{l}_2^{-1}. \hspace{2cm} (5.21)$$

We show that it gives another possibility of trace factorisation. Starting from (4.8) and using (5.21) and (5.20) we get

$$R_{12}(u - v)\hat{l}_1^{-1}T_1(u)\hat{l}_1\hat{l}_2^{-1}T_2(v)\hat{l}_2 = \hat{l}_2^{-1}T_2(v)\hat{l}_2\hat{l}_1^{-1}T_1(u)\hat{l}_1R_{12}(u - v).$$  \hspace{2cm} (5.22)

Taking trace of (5.22) results in the required commutation relation $[\text{tr}\hat{T}(u), \text{tr}\hat{T}(v)] = 0$, where $\hat{T}(u) = \hat{l}^{-1}\hat{T}(u)\hat{l}$

## 6 Quantum nonultralocal models

For application to integrable models we may take any of the above ansätze for $Z$ ensuring the trace factorisability, which is a basic requirement of integrability and look for different solutions of (2.15) and (3.9)–(3.10) for a given $R$-matrix. It is important to note that the factorisation procedure does not touch the $\tilde{Z}$-matrix allowing it to be any arbitrary solutions of (2.16)–(2.18) with additional constraint (4.7) in case of periodic chains. We show below that the nonultralocal quantum models including supersymmetric models can be explained using the present framework for the simplest choice $\tilde{Z} = 1$ or $\tilde{Z} = Z$. These examples are associated with different solutions for $Z$ and $R$-matrices. It is curious to note that all examples known to us, including those related to CFT, are in agreement with above forms of $Z$ leading to trace factorisability.

Since Hecke condition underlies our construction we restrict to the $R_q^\pm$ matrices of $U_q(sl(n|m))$ with $q = e^{i\eta}$ leading to the trigonometric solution

$$R_{12}(u) = \sum_{\alpha, \beta} (\sin u e_{\alpha\beta} \otimes e_{\beta\alpha} + \epsilon_\alpha \sin(u + \epsilon_\beta) e_{\alpha\alpha} \otimes e_{\alpha\alpha} + \sin \eta e_{\alpha\beta} \otimes e_{\beta\alpha})$$  \hspace{2cm} (6.1)

where $\epsilon_\alpha = \pm 1$ are obtained from the Baxterisation of 'standard' or 'nonstandard' solutions, respectively. At $q \to 1$ (6.1) reduces to the rational solution of YBE. Note
that this type of $R$-matrix was proposed by Perk and Schultz [20] in connection with integrable statistical models. Such $R$-matrices exhibit a symmetry

$$[R_{12}(u), s_1 s_2] = 0, \quad s = \sum_\alpha \epsilon_{\alpha \alpha} \phi_\alpha \quad (6.2)$$

Therefore if $s$ commutes with $Z$, along with the usual trace $tr T$ one also gets the commutativity of ‘$s$-trace’ $tr (sT)$ . This property is used for example, in the construction of SUSY models for physical reasons.

We look now into some concrete examples of quantum nonultralocal models known in the present day literature, which were proposed over the years in various contexts and show that in the framework of the present theory based on the quantised braided algebra it is possible to describe them in an unifying way.

### 6.1 Supersymmetric models

Quantum integrable supersymmetric theory [21, 22] is possible to formulate in a convenient alternative way expressing the gradings in a matrix form as proposed in [16]. We observe that such a formulation can be reproduced nicely within our general framework and corresponds to case 1 in the previous section, where now $Z = \tilde{Z} = \eta$ and $\eta$ is in the form (5.12) with $g_{\alpha \beta} = (-1)^{\hat{\alpha} \beta}$. Here $\hat{\alpha}$ describes the supersymmetric grading with $\hat{\alpha} = 0(1)$ depending on the even (odd)-ness of the index. The quantum integrability follows from the commuting traces due to (5.11), though however for physical reasons ‘supertraces’ are usually used for such models, which may be obtained from the ‘$s$-trace’ (6.2) by choosing $\phi_\alpha = (-1)^{\hat{\alpha}}$. The corresponding super QYBE’s for these SUSY models should be derived from (4.1),(4.8) as

$$R_{12}(u - v)\eta_{12} L_{1j}(u) \eta_{12} L_{2j}(v) = \eta_{12} L_{2j}(v) \eta_{12} L_{1j}(u) R_{12}(u - v) \quad (6.3)$$

$$R_{12}(u - v)\eta_{12} T_1(u) \eta_{12} T_2(v) = \eta_{12} T_2(v) \eta_{12} T_1(u) R_{12}(u - v) \quad (6.4)$$

while from the nonultralocal braiding relations (4.3) and (4.4) one obtains the super-commutation relations as

$$L_{2k}(v) \eta_{12} L_{1j}(u) = \eta_{12} L_{1j}(u) \eta_{12} L_{2k}(v) \eta_{12} \quad (6.5)$$

for $k \neq j$. In order that these equations may be compared with the similar ones appearing in the literature on SUSY models [21, 22] we express (6.3) in the matrix element form

$$R_{a_1 a_2}^{b_1 b_2}(u - v)(L_{b_1}^{c_1}(u))_j (L_{b_2}^{c_2}(v))_j (-1)^{\hat{b}_2 (\hat{b}_1 + \hat{c}_1)} = (-1)^{\hat{a}_1 (\hat{a}_2 + \hat{b}_2)} (L_{a_2}^{b_2}(v))_j (L_{a_1}^{b_1}(u))_j R_{b_1 b_2}^{c_1 c_2}(u - v) \quad (6.6)$$

and the braiding relation (6.5) as

$$(L_{a_2}^{b_2}(u))_k (L_{a_1}^{b_1}(v))_j = (-1)^{\hat{a}_1 (\hat{a}_2 + \hat{b}_2) (\hat{b}_2 + \hat{b}_2)} (L_{a_2}^{b_1}(u))_j (L_{a_1}^{b_2}(v))_k \quad (6.7)$$

Note that in contrast to nonultralocal models discussed below different braiding relations coincide here showing ‘long range nonultralocality’ of SUSY models. Choosing
the $R$-matrix as rational or trigonometric solutions we can derive the results of [21] and [22], respectively.

An immediate generalisation of SUSY models to their braided analogs can be done by putting $\bar{Z} = Z$, where $Z$ is a solution of (2.15), (3.9) and (3.10) that enables trace factorisation like for example in the cases discussed in the previous section.

We stress again that for supersymmetric models or their braiding analogs we limit ourselves to $\bar{Z} = Z$ that exhibits 'homogeneous' long range nonultralocality, whereas for inclusion of the following models with only nearest-neighbour nonultralocality we need $Z \neq \bar{Z}$.

### 6.2 Nonabelian Toda chain

This is a one-dimensional evolution model on a periodic lattice of $N$ sites described by the nonabelian matrix-valued operator $g_k \in GL(n)$. The model, representing a discrete analog of principal chiral field, was set into the Yang-Baxter formalism in [4]. The Lax operator of this nonultralocal quantum model was presented as

$$L_k(\lambda) = \begin{pmatrix} \lambda - A_k & -B_{k-1} \\ I & 0 \end{pmatrix}, \quad A_k = \hat{g}_k g_k^{-1}, \quad B_k = g_{k+1}^{-1} \quad (6.8)$$

and the associated $R$-matrix is the rational solution of YBE $R(\lambda) = i\hbar P - \lambda$ with $P = \Pi \otimes \pi$, where $\Pi$ is a $4 \times 4$ and $\pi$ a $n^2 \times n^2$ permutation matrix. We observe that this model can be described nicely by our formalism by choosing the related $Z, \bar{Z}$ matrices as $\bar{Z} = 1$ and $Z_{12} = 1 + i\hbar (e_{22} \otimes e_{12}) \otimes \pi$. Moreover we notice that this is an example of the case discussed in subsection 6.2, since by choosing $\hat{I} = diag(I, gn)$ and using the explicit form (6.8) of $L_i(u)$, one can show [4] that $Z_{21}^{-1} L_{1N} = \hat{I}_2 L_{1N} \hat{I}_2^{-1}$ leading to the requirement (5.21): $Z_{21}^{-1} T_1 = \hat{I}_2 T_1 \hat{I}_2^{-1}$. The validity of other conditions (5.20) can also be readily checked. Therefore the quantum integrability of the model is evident from trace identity (5.22). With this input one finds easily that our main formulas (4.1) and (4.8) recover exactly the QYBE’s of [4], while the nonultralocal braiding relations should be given by

$$L_{2j+1} Z_{21}^{-1} L_{1j}(u) = L_{1j}(v) L_{2j+1}(v), \quad L_{2k}(v) L_{1j}(u) = L_{1j}(u) L_{2k}(v) \quad (6.9)$$

for $k > j + 1$.

Now we look into lattice regularised models related to the conformal field theory, which were proposed to describe current and exchange algebras in the WZNW [7] and in Toda field theory [6]. Since the approach of [7] and [6] is not restricted by the demand of integrability, we may limit ourselves to the spectral parameter independent equations and in principle consider $Z$ without having the trace factorisation property discussed in sect. 5.

### 6.3 Quantum symmetry in WZWN model

The WZWN model is described by the unitary unimodular matrix-valued field $g(x,t) : M \to su(2), \quad M = S^1 \times R^1$. Defining the chiral left current as $L = \frac{1}{2} (J_0 + J_1)$, where
\( J_u = \partial_u g g^{-1} \), one gets the well known current algebra relation in the form of Poisson bracket

\[
\{ L_1(x), L_2(y) \} = \frac{\gamma}{2} [C, L_1(x) - L_2(y)] \delta(x - y) + \gamma C \delta'(x - y)
\]

where \( C_{12} = 2P_{12} - 1 \), giving the Kac-Moody algebra with the central charge defined by the coupling constant \( \gamma \). Introducing \( \partial_u u = L_u \) one can write the monodromy as \( M_L = u(x)^{-1} u(x + 2\pi) \) and the similar relations for the right current. With the aim of unravelling the quantum group structure in WZWN model Faddeev and his collaborators [7] formulated a discrete and quantum version of the current algebra, where \( L_j \) now stands for the lattice regularised current in the WZWN model and the chiral component becomes \( u(x) \rightarrow u_j \). The nonultralocal current algebra at this discrete level was shown [7] to be equivalent to the commutation relations

\[
R_{i12}^+ L_{1j} L_{2j} R_{i12}^- = L_{2j} L_{1j} \tag{6.10}
\]

\[
L_{2j+1} R_{i12}^+ L_{1j} = L_{1j} L_{2j+1}, \quad L_{2k} L_{1j} = L_{1j} L_{2k} \tag{6.11}
\]

for \( k > j + 1 \), while the algebraic relations for the chiral components were represented by

\[
R_{i12}^+ u_{1j} u_{2j} R_{i12}^- = u_{2j} u_{1j} \tag{6.12}
\]

\[
u_{1j} u_{2n} R_{i12}^+ = u_{2n} u_{1j} \tag{6.13}
\]

for \( j > n \) and for the monodromy as

\[
M_{L1}(R_{i12}^-)^{-1} M_{L2} R_{i12}^- = (R_{i12}^+)^{-1} M_{L2} R_{i12}^+ M_{L1} \tag{6.14}
\]

We show that all these results can be reproduced from the general scheme presented here. For this we set \( \tilde{Z} = 1 \) and \( Z_{i2} = R_{i12}^- \) and note that this is the solution considered above as case 4 and agrees with our ansatz d). Further we identify \( L_{aj} \) of [7] with our \( L_{aj}^+ \) and define the analogous ‘monodromy’ matrices \( T^{[j,k]} \) and \( T \) as in (4.2) and (1.3). From (5.16) with \( \tilde{Z} = 1 \) we then get

\[
R_{i12}^+ L_{1j} L_{2j} R_{i12}^- = L_{2j} L_{1j} \tag{6.15}
\]

the equivalence of which with the current algebra relation (6.10) is evident. Similarly our braiding relations (5.17) recover the nonultralocal relation at different points (6.11). Identifying \( u_j = T^{[j,3]} \) and \( M \) with \( T \) we obtain the commutation relations for chiral components (6.12) as well as for the monodromy (6.14).

Let us remark that in this specific case we can derive also the commutation relation between \( T^{[j,k]} \) and \( T^{[n,k]} \):

\[
\tilde{Z}_{12} T^{[j,k]} \tilde{Z}_{21} T^{[n,k]} R_{i12}^+ = T^{[n,k]} \tilde{Z}_{12} T^{[j,k]} \tag{6.16}
\]

for \( j > n \). Putting \( \tilde{Z} = 1 \) in the above equation we get the exchange relations for the chiral components (6.13).
6.4 Quantum group structure in Coulomb gas picture of CFT

The Coulomb gas picture of CFT is based on the Drinfeld-Sokolov linear system [6] \( \partial Q = L(x)Q \), with the Lax operator

\[
L(x) = P(x) - \mathcal{E}_+, \quad \mathcal{E}_+ = \sum_{\alpha \text{ simple roots}} E_\alpha.
\]

The periodic field \( P(x) \) satisfies the Poisson bracket relation

\[
\{P(x) \otimes P(y)\} = \delta'(x - y) \sum_i H_i \otimes H_i
\]
causing the nonultralocal nature of the model. In [6] a lattice regularised description of the algebraic structures was given at the quantum level, where the commutation relation of the discretised Lax operator was presented in the form

\[
R_{12}^+ \tilde{L}_{1j} \tilde{L}_{2j} = \tilde{L}_{2j} \tilde{L}_{1j} R_{12}^+
\]
and

\[
\tilde{L}_{2j+1} A_{12} \tilde{L}_{1j} = \tilde{L}_{1j} \tilde{L}_{2j+1}, \quad \tilde{L}_{2k} \tilde{L}_{1j} = \tilde{L}_{1j} \tilde{L}_{2k}, \quad k > j + 1
\]
where \( A_{12} = q \sum H_i \otimes H_i, \quad q = e^{\frac{i}{2} \hbar}. \) From these basic starting relations the algebras of monodromies were found as

\[
R_{12}^+ Q_{1j} Q_{2j} = Q_{2j} Q_{1j} R_{12}^+
\]
for \( Q_{j} = \tilde{L}_{j} B \tilde{L}_{j-1} B \ldots B \tilde{L}_{3} \) with \( B_3 = q^{\frac{1}{2}} \sum \alpha \mathcal{R}_\alpha^2 \) and for the periodic lattice of \( N \) sites with monodromy \( S = \tilde{L}_N B \tilde{L}_{N-1} B \ldots B \tilde{L}_3 \) as

\[
R_{12}^+ S_1 A_{12} S_2 = S_2 A_{12} S_1 R_{12}^+
\]

For relating this structure with our formulation we must consider again the spectral parameter independent case as in the previous example and choose \( Z = 1 \) and \( Z_{12} \equiv (A_{12})^{-1}. \) This corresponds to case 5 of sect. 5.1 and we identify \( B \) with \( k^{\frac{1}{2}}. \) The relations (6.17) and (6.18) are then recovered directly from (6.19) and the braiding equations (2.12–2.13). For obtaining the monodromy relations we define \( Q_j \equiv B^{-1} T^{[j;3]} \) and \( S \equiv B^{-1} T \) and repeating the steps of the previous example derive the relations (6.19) and (6.20).

6.5 Nonultralocal quantum mapping

The quantum mapping associated with the lattice Gelfand-Dikii hierarchy is given by the Lax operator \( L_n \) in the form [5] \( L_n = V_{2n} V_{2n-1} \), where

\[
V_n = \Lambda_n \left( 1 + \sum_{i>j=1}^N v_{i,j}(n) e_{i,j} \right).
\]
The commutation relation of the hermitean operators \( v_{i,j} \) is

\[
[v_{i,j}(n), v_{k,l}(m)] = \hbar (\delta_{n,m+1}\delta_{k,j+1}\delta_{i,l} - \delta_{m,n+1}\delta_{i,l+1}\delta_{k,j,1})
\]

which acquires nontrivial values also at nearest-neighbour lattice points. This clearly makes the corresponding Lax operator nonultralocal in nature. The related \( R \)-matrix is given by the rational solution

\[
R_{12}(u_1 - u_2) = 1 + \frac{\hbar P_{12}}{u_1 - u_2}.
\]  

(6.21)

It has been found in [5] that the quantum Yang-Baxter equation for this model may be given by

\[
\tilde{R}_{12}(u_1 - u_2)L_{1j}(u_1)L_{2j}(u_2) = L_{2j}(u_2)L_{1j}(u_1)R_{12}(u_1 - u_2),
\]  

(6.22)

along with nonultralocal conditions

\[
L_{2j+1}(u_2)S_{21}(u_1)L_{1j}(u_1) = L_{1j}(u_1)L_{2j+1}(u_2)
\]  

(6.23)

and

\[
L_{2k}(u_2)L_{1j}(u_1) = L_{1j}(u_1)L_{2k}(u_2),
\]  

(6.24)

for \( k > j + 1 \). In the above formulas

\[
\tilde{R}_{12}(u) = (S_{12}(u))^{-1}R_{12}(u_1 - u_2)S_{21}(u_1) \quad \text{and} \quad S_{12}(u_2) = 1 - \frac{\hbar}{u_2} \sum_{\alpha} e_N \otimes e_{\alpha N}
\]  

(6.25)

The equation for the monodromy matrix with the periodic boundary condition is of the form

\[
\tilde{R}_{12}(u_1 - u_2)T_1(u_1)S_{12}(u_2)T_2(u_2) = T_2(u_2)S_{21}(u_1)T_1(u_1)R_{12}(u_1 - u_2).
\]  

(6.26)

This model actually goes beyond the scope of our formalism, because the matrix \( S_{12} \) depends on the spectral parameter and besides that it does not satisfy the YBE (2.15). Nevertheless, if we put formally \( \tilde{Z} = 1 \) and \( Z_{12} = S_{12}^{-1}(u_2) \), then we can see that \( \tilde{R} \) is of the form (2.21), which was mentioned in section 2 as an alternative to (2.15). Morerover this choice fits into our ansatz (5.8) and satisfies the restriction (5.11). Therefore the relevant equations can be derived from (5.14)-(5.15).

7 New nonultralocal quantum models through application of braided structures

For showing the usefulness of our scheme based on the braiding relation and the braided quantum Yang–Baxter equation, we present in this section some new examples of nonultralocal quantum models constructed from them.
7.1 Quantum mKdV model

Modified KdV equation

\[ \pm v_t + v_{txx} + 12v^2v_x = 0 \]  

(7.1)

is a well known classical integrable system [23] with wide range of applications [24]. This model possesses an infinite set of conserved quantities \( c_n, \ n = 1, 2, \ldots \) at the classical level and exhibits noncanonical Poisson bracket structure

\[ \{v(x), v(y)\} = \mp 2\delta'(x - y). \]  

(7.2)

(7.1) may be derived as a Hamilton equation from

\[ H = c_3 = \frac{1}{2} \int_{-\infty}^{\infty} dx \left( v_x^2 + v^4 \right) \]

by using (7.2). However, as it is known, due to its nonultralocal algebraic structure (7.2) the much awaited quantum generalisation of this model could not be achieved through usual procedure [1, 2].

We propose a new quantum version of the mKdV model and show that its exact integrability at the lattice level can be established through our scheme built on the braided quantum algebraic structures. For this purpose we perform lattice regularisation by constructing the quantum Lax operator of the discrete model as

\[ L_k(\zeta) = \begin{pmatrix} e^{-\frac{i}{2}v_k^-} & \frac{\Delta \zeta}{2} e^{\frac{i}{2}v_k^+} \\ -\frac{\Delta \zeta}{2} e^{-\frac{i}{2}v_k^+} & e^{\frac{i}{2}v_k^-} \end{pmatrix}, \]  

(7.3)

with the commutation relations between the operators \( v^\pm_k \) as

\[ [v^+_k, v^\pm_l] = \mp i\hbar(\delta_{k-1,l} - \delta_{k,l-1}), \]  

(7.4)

\[ [v^+_k, v^-_l] = i\hbar(\delta_{k-1,l} - 2\delta_{k,l} + \delta_{k,l-1}). \]  

(7.5)

Note that at the continuum limit, i.e. at \( \Delta \to 0 \), the discrete operators \( v^\pm_k \) go to the quantum field \( \Delta v^\pm(x) \) and the relation (7.4) reduces to the nonultralocal commutation relation

\[ [v^\pm(x), v^\pm(y)] = \pm i\hbar(\delta_x(x - y) - \delta_y(x - y)). \]  

(7.6)

It may be puzzling at the first sight that the number of fields has become double. Therefore we should note that this pair of similar fields simply answer to the different possible signs of the mKdV eqn. (7.1) and the dependence of one of them drops out from the Lax operator (7.3) at the continuum limit:

\[ L_k(\zeta) \to \mathbf{1} + \Delta \mathcal{L}(x, \zeta) + O(\Delta^2) \]

recovering the known Lax operator of the continuum model:

\[ \mathcal{L}(x, \zeta) = \frac{i}{2}(-v^{-}(x)\sigma^3 + \zeta\sigma^2) \]  

(7.7)
corresponding to (7.1) with + sign. For getting the other sign the role of \( v_k^\pm \) should be interchanged in (7.3). A simple similarity transformation \( A^{-1} \mathcal{L}(x, \zeta) A = \mathcal{L}_{mkdv}(x, \zeta) \) gives further the Lax operator of the mKdV model in the well known AKNS form \( \mathcal{L}_{mkdv}(x, \zeta) = \frac{i}{2}(\zeta \sigma^3 - v(x) \sigma^1). \)

We show now that for finding the quantum \( R \)-matrix and the quantum Yang–Baxter equation associated with the mKdV Lax operator (7.3) we may start with the braiding relation (4.3). Using (7.4), (7.5) and assuming \( \tilde{Z} = 1 \) and \( \zeta = e^{i\lambda}, \eta = e^{i\mu} \) we find from

\[
L_{2j}(\lambda) L_{1j+1}(\mu) = L_{1j+1}(\mu) Z_{12}^{-1} L_{2j}(\lambda)
\]

the matrix

\[
Z_{12} = Z_{21} = q^{-\frac{1}{2}} e^{\gamma \sigma^3} \sigma^3 ,
\]

where \( q = e^{i\gamma} \). Plugging this \( Z \)-matrix in the braided QYBE (4.1) and using relations like

\[
[e^{i\alpha v^+_j}, e^{i\beta v^-_j}] = 0, \quad e^{i\alpha v^+_j} e^{i\beta v^-_j} = e^{i\alpha \beta} e^{i\beta v^-_j} e^{i\alpha v^+_j}
\]

we derive after some algebra the quantum \( R \)-matrix of this model, which turns out to be the standard trigonometric solution (6.1) for \( m + n = 2 \):

\[
R(\lambda - \mu) = \sum_{\alpha, \beta=1}^{2} \left( \sin(\lambda - \mu) e_{\alpha \alpha} \otimes e_{\beta \beta} + \sin((\lambda - \mu) + \frac{\hbar}{2}) e_{\alpha \alpha} \otimes e_{\alpha \alpha} + \sin(\frac{\hbar}{2}) e_{\beta \alpha} \otimes e_{\beta \alpha} \right)
\]

(7.10)

It is an easy check that (7.10) and (7.9) thus found satisfy the prescribed consistency equations (3.8–3.10) and (2.15). Therefore, we may use these \( R, Z \)-matrices to find the braided QYBE for the monodromy matrix of the periodic quantum mKdV model in the form (4.8). Moreover, due to the fact that the \( R \)-matrix (7.10) commutes with the \( Z \)-matrix (7.9) the braided QYBE’s in the present case simplify further to

\[
R_{12}(\lambda - \mu) L_{1j}(\lambda) L_{2j}(\mu) = L_{2j}(\mu) L_{1j}(\lambda) R_{12}(\lambda - \mu)
\]

(7.11)

and

\[
R_{12}(\lambda - \mu) T_{1}(\lambda) Z_{12}^{-1} T_{2}(\mu) = T_{2}(\mu) Z_{12}^{-1} T_{1}(\lambda) R_{12}(\lambda - \mu),
\]

(7.12)

respectively. For identifying the commuting set of conserved quantities from the trace identity, one may define a operator valued matrix \( \hat{l} = e^{-\frac{i}{2}q_N \sigma^3} \), with \( q_N \) canonical to \( v_N^\pm : [q_N, v_N^\pm] = \pm i\hbar \), so that \( \hat{T}(\lambda) = \hat{l}^{-1} T(\lambda) \hat{l} \). Following the idea of sect. 5.2 one shows that \( [tr \hat{T}(\lambda), tr \hat{T}(\mu)] = 0 \) and thus establishes the integrability relations of this nonultralocal model at the quantum level. A more detailed account of this model including the Bethe ansatz solution will be given elsewhere.

## 7.2 Quantum integrable anyonic supersymmetric model

We have seen in sect. 6.2 that the SUSY models are covered by the nonultralocal integrability scheme developed here. In this subsection we propose background for a new quantum integrable anyonic generalisation of the SUSY models and show that it can be constructed in a systematic way following the same formalism.
In analogy with the standard SUSY model we take $Z = \bar{Z}$ but choose its explicit form in a more general way:

$$Z_{12} = \sum_{\alpha, \beta} e^{i\theta} \hat{a}_\alpha \hat{b}_\beta e_{\alpha\alpha} \otimes e_{\beta\beta}$$

(7.13)

where $\theta$ is the arbitrary anyonic phase. For such a $Z$-matrix the braiding relation describing the commuting relations between elements of the Lax operator at different lattice points $k > j$ take the form

$$L^a_{b_2(k)}(\mu)L^a_{b_1(j)}(\lambda) = e^{-i\theta(\hat{a}_1 - \hat{b}_1)(\hat{a}_2 - \hat{b}_2)} L^{a_1}_{b_1(j)}(\lambda)L^{a_2}_{b_2(k)}(\mu),$$

(7.14)

with all matrix indices running from 1 to $N = m + n$. The algebraic relations at the same point $l$ on the other hand are given by the corresponding braided QYBE

$$\sum_{\{k\}} R^{k_1k_2}_{a_1a_2}(\lambda - \mu)L^b_{k_1(l)}(\lambda)L^b_{k_2(l)}(\mu)e^{i\theta k_2(\hat{b}_1 - \hat{k}_1)} = \sum_{\{k'\}} e^{i\theta \hat{a}_1(\hat{k}_2' - \hat{a}_2)} L^{k'_2}_{a_2(l)}(\mu)L^{k_1}_{a_1(l)}(\lambda) R^{k_1k_2}_{a_1a_2}(\lambda - \mu),$$

(7.15)

The quantum $R$-matrix may be chosen in the trigonometric form (6.1) or as its rational limit

$$R_{12}(\lambda) = E_{12} \lambda + \frac{\hbar}{2} P_{12},$$

(7.16)

where $P$ is the permutation matrix and $E = I$ for the 'standard' and $E_{12} = \eta_{12}$ for the 'nonstandard' solutions. The consistency equations for $Z$ and $R$ matrices obviously hold for all these solutions. Taking the Lax operator of the associated models related to the rational case as

$$L^b_{a(l)}(\lambda) = \lambda \delta_{ab} p^0_b + \frac{\hbar}{2} e^{i\theta(\hat{a}_1)(\hat{b}_1)} p^0_{ba}$$

(7.17)

and using the rational $R$-matrix solution (7.16) one may derive from (7.15) the anyonic super algebra (ASA) at the same lattice point by matching the coefficients of different powers of the spectral parameter. This yields the following set of algebraic relations between the generators of this graded algebra.

$$p_{b_2a_2}^{(l)} p_{b_1a_2}^{(l)} - e^{i\theta(\hat{a}_2 - \hat{b}_1 \hat{a}_2)} p_{b_2a_2}^{(l)} p_{b_1a_2}^{(l)} = e^{-i\theta(\hat{a}_2 \hat{a}_1)} \delta_{a_2b_2} p_{b_1a_2}^{(l)} p_{b_2a_2}^{(l)} - e^{i\theta(\hat{a}_2 - \hat{b}_2)(\hat{a}_2 + \hat{b}_2)} \delta_{a_2b_2} p_{b_1a_2}^{(l)} p_{b_2a_2}^{(l)}$$

(7.18)

and

$$p_{a_1}^{(l)} p_{b_2a_2}^{(l)} = e^{i\theta(\hat{a}_2 - \hat{b}_2)} p_{b_2a_2}^{(l)} p_{a_1}^{(l)}$$

(7.19)

for the standard $R$-matrix. In the nonstandard case one obtains another type of ASA, where additional multiplicattive factors $e^{i\pi(\hat{a}_1 \hat{a}_2)}$ and $e^{i\pi(\hat{b}_1 \hat{b}_2)}$, appear before the first and the second term of algebra (7.18), respectively, while the relation (7.19) turns into

$$p_{a_1}^{(l)} p_{b_2a_2}^{(l)} = e^{i(\theta + \pi)(\hat{a}_2 - \hat{b}_2)} p_{b_2a_2}^{(l)} p_{a_1}^{(l)}$$

(7.20)

In the nonstandard case, as can be easily checked, we obtain also an extra relation

$$(p_{ab}^{(l)})^2 = 0, \quad \text{for} \quad \hat{a} - \hat{b} = 1,$$

17
reflecting the fermionic type character. We use braiding relation (7.14), for obtaining the algebra at different lattice points:

\[
P_{a_2 a_3}^{(k)} P_{a_1 b_1}^{(j)} = e^{-i\theta (a_1 - \bar{a}_1)(a_2 - \bar{a}_2)} P_{a_1 b_1}^{(j)} P_{a_2 a_3}^{(k)}
\]

(7.21)

for \( k > j \) with all other commutators being trivial. This relation remains same in both standard as well as nonstandard cases, since the braiding relation does not involve \( R \)-matrix solutions.

Note that this algebra is a nontrivial generalisation of the well known graded super algebra [21], where together with an arbitrary anyonic parameter \( \theta \) a set of \( N \) additional operators \( p_a^{(0)} \) appear. For recovering the known super algebra we have to start with our relations corresponding to the nonstandard \( R \)-matrix and choose \( \theta = \pi \). It is easily seen from (7.20) that for such a choice \( p_a^{(0)} \) commute with all other operators and hence may be chosen as unity.

For constructing quantum integrable models involving \( m \) number of bosons \( b^{(k)}_a \) and \( n \) number of anyons \( f^{(k)}_a \), as a direct generalisation of the integrable SUSY models, we may consider a realisation of the generators as

\[
p_{a b}^{(k)} = b^{+(k)}_a b^{(k)}_b, \quad p_{a a}^{(k)} = f^{+(k)}_a f^{(k)}_a, \quad p_{a a}^{(k)} = b^{+(k)}_a f^{(k)}_a, \quad p_{a a}^{(k)} = f^{+(k)}_a f^{(k)}_a.
\]

(7.22)

The bosons satisfy the standard commutation rules and they commute with all anyons, while the anyonic operators have the following commutation relations involving the phase factor \( \theta \).

\[
f^{(k)}_\alpha f^{+(j)}_\beta = e^{i \theta} f^{+(j)}_\beta f^{(k)}_\alpha, \quad (7.23)
\]

\[
f^{(k)}_\alpha f^{(j)}_\beta = e^{-i \theta} f^{(j)}_\beta f^{(k)}_\alpha, \quad (7.24)
\]

for \( k > j \) and

\[
[p_a^{(0)}, f^{(j)}_\beta] = 0, \quad (7.25)
\]

for \( k \neq j \), while

\[
f^{(l)}_\alpha f^{+(l)}_\beta - e^{i \theta} f^{+(l)}_\beta f^{(l)}_\alpha = \delta_{\alpha \beta} p_a^{(0)}
\]

(7.26)

and

\[
p_a^{(0)} f^{(l)}_\beta = e^{-i \theta} f^{(l)}_\beta p_a^{(0)}, \quad [f^{(l)}_\alpha, f^{(l)}_\beta] = 0,
\]

(7.27)

at the same lattice points. Note that these relations are consistent with the anyonic algebra well known in the literature [25, 26]. The existence of such anyonic operators in one dimension is also in accordance with [27], where the existence of anyons has been established in any dimensions. It is important to note that though one can take \( f^{+(k)}_\alpha = (f^{(k)}_\alpha)^\dagger \) to derive other commutation relations for \( k \neq j \), it seems not to hold at the same lattice points with nontrivial \( \theta \). This kind of problem arising at the coinciding points also known to exist in the standard anyonic theory [28, 26]. The associated anyonic SUSY integrable models should be represented by the Lax operator (7.17) along with realisations like (7.22). An explicit construction of such a model and its exact solution will be presented in a future publication [29].
7.3 \textit{q}-deformed anyonic super algebra

We may generate a novel \textit{q}-deformation of the above ASA (7.18)-(7.21) by simply taking the same Z-matrix (7.13) and choosing the trigonometric \textit{R}-matrix solution (6.1). Representing the Lax operator through the corresponding generators $\tau^{\pm}_a, \tau^{(l)}_{ab}$ of the algebra as

$$L^b_{a(l)}(\lambda) = \delta_{ab} (e^{i\lambda \tau^{+}_b} + e^{-i\lambda \tau^{-}_b}) + e^{i\theta \delta \tau^{-}_b \tau^{-}_b}$$

and comparing the coefficients of different spectral parameters from (7.15) and (7.14) we derive the set of relations at coinciding as well as different points forming the \textit{q}-ASA. For example, a set of such relations in the general form for arbitrary $m, n$ grading looks like

$$e^{-i\delta(\delta_1 \delta_2)} (f(a_1) - 1) \delta_{a_1a_2} + 1 \right) \tau^{+}(l) \tau^{+}(l) + (q - q^{-1}) e^{-i\delta(\delta_2 \delta_2) \tau^{(l)}_{c_1a_1} \tau^{+}(l) \tau^{+}(l) (a_1 < a_2)

= e^{-i\delta(\delta_2 \delta_2)} (f(c_1) - 1) \delta_{c_1c_2} + 1 \right) \tau^{+}(l) \tau^{+}(l) + (q - q^{-1}) e^{-i\delta(\delta_2 \delta_2) \tau^{(l)}_{c_1a_1} \tau^{+}(l) \tau^{+}(l) (c_2 < (c_4))$

and

$$\tau^{+}(k) \tau^{+}(j) = e^{-i\delta(\delta_1 \delta_1) \delta_2 \delta_2) \tau^{(l)}_{c_1a_1} \tau^{+}(l) \tau^{+}(l)$$

where we have denoted $\tau^{+}(j) = \tau^{(j)}_a, \tau^{+(j)}_{c_1a_1} = \tau^{(j)}_{c_1a_1}$ for $c > a$ and $f(a) = 1$ for the standard \textit{R}-matrix, while $f(a) = 1 + \delta(\delta_1 \delta_1 - 2)$ for the nonstandard solution.

For simplicity we present here the $m = 1, n = 1$ case corresponding to the standard trigonometric \textit{R}-matrix solution in the following explicit form.

$$\tau^{(l)}_{21} \tau^{(l)}_{21} = e^{-i\delta(\delta_1 \delta_1) \delta_2 \delta_2) \tau^{(l)}_{c_1a_1} \tau^{+}(l) \tau^{+}(l)$$

$$\tau^{\pm}(l) \tau^{\pm}(l) = q^\pm e^{i\delta \tau^{(l)}_{c_1a_1} \tau^{+}(l) \tau^{+}(l) \tau^{(l)}_{c_1a_1} \tau^{+}(l) \tau^{+}(l)$$

at the same lattice point $l$, with all $\tau^{\pm}(l)$ mutually commuting and $a, b = 1, 2$. The nontrivial commutation relations at different points are given by

$$\tau^{(l)}_{12} \tau^{(l)}_{21} = e^{i\delta \tau^{(l)}_{21} \tau^{(l)}_{12} \tau^{(l)}_{c_1a_1} \tau^{+}(l) \tau^{+}(l)$$

for $k > j$. This super algebra represents a generalisation of the extended trigonometric Sklyanin algebra [14] to include an anyonic parameter $\theta$ and nonultralocal algebraic property (7.32). This algebra would naturally lead to a novel anyonic quantum group with 'long range' nonultralocality. One can recover the super $sl_q(m,n)$ algebra introduced in [22] by choosing $\theta = \pi$ and considering the nonstandard \textit{R}-matrix. Realisation of this algebra through anyonic quantum group as well as through a new anyonic $q$-oscillator is possible. Detailed discussion of this problem along with concrete model construction will be given elsewhere [29].

8 Conclusion

We have presented a scheme suitable for describing quantum nonultralocal models including supersymmetric models. The scheme is based on the concept of quantised braided groups. We have noticed that while the braiding of supersymmetric models
is 'uniform', i.e. the commutation relations of spatially separated objects remains the same independent of their distance, the commutation relations in nonultralocal models exhibit different properties for near and distant neighbours.

For that reason we generalise the quantised braided groups to include different types of braiding to distinguish between nearest and nonnearest neighbours, and express it in the Faddeev–Reshetikhin–Takhtajan algebra form. To frame the formalism for application to integrable models we introduce spectral parameter through Baxterisation of the Faddeev–Reshetikhin–Takhtajan relations. This results in a braided quantum Yang–Baxter equation along with nonultralocal commutation relations initiated by the braiding. Due to the nonultralocal commutation relations, the underlying coalgebra structure remains intact and that in turn enables us to construct monodromy matrices satisfying braided quantum Yang–Baxter equation both for periodic and finite lattices.

In contrast to ultralocal models, the problem of deriving a set of commuting operators from quantum Yang–Baxter equation becomes nontrivial. Nevertheless we are able to solve it for several classes of braiding.

We show the scope of our formalism by describing the supersymmetric as well as nonultralocal quantum models proposed in the context of integrable or conformal field theory, as different examples of the theory presented here. The last known example we consider does not fit completely in our scheme and suggests some possibilities of its further generalisation to spectral parameter dependent braiding.

As further applications of the proposed scheme we are able to find new examples of nonultralocal models as quantum integrable systems. Following our formalism we have solved the important problem of describing the well known classically integrable nonultralocal mKdV model at the quantum level. We have shown the possibility of constructing a new class of SUSY models involving bosonic as well as anyonic fields. Finally we find a novel quantum deformation of the anyonic super algebra with arbitrary grading.

It would be interesting also to explore various other possibilities by taking the braiding as suggested in sect. 6.1, apart from the long-standing problems of quantising nonultralocal models like nonlinear σ-model, derivative nonlinear Schrödinger equation, sine-Gordon equation in light-cone coordinates etc. For inclusion of wider class of nonultralocal models, e.g. KdV model in its second Hamiltonian formulation related to the Virasoro algebra, it seems that one would need braided structures containing more than two types of braiding. This offers another direction for extension of the present formalism.

Acknowledgement
One of the authors (AK) acknowledges the support of Alexander von Humboldt Foundation research fellowship grant and expresses his thanks to Prof. Vladimir Rittenberg, Fabian Essler and other members of the Theory Division of the Physikalisches Institut, Bonn for stimulating discussions.

L. Hlavaty would like to express his gratitude to Physikalisches Institut, Bonn for hospitality during the preparation of the final version of the paper and acknowledges the support of the grant No. 202/93/1314 of the Czech Republic and the grant No.8154
References


[16] Li Liao, X-Ch. Song, Mod. Phys. Lett. 6 (1991) 959.


[29] A. Kundu under preparation

22