EVALUATION REPORT ON DR. ONDŘEJ TUREK'S HABILITATION THESIS

The Habilitation Thesis by Dr. Ondřej Turek is devoted to quantum graphs - ordinary differential operators on metric graphs - the area he has been working in since his PhD studies with P. Exner, one of the experts in the area. One can clearly see two major themes discussed:

- vertex conditions and symmetries;
- infinite periodic quantum graphs in connection with Bethe-Sommerfeld conjecture.

It is natural to discuss these two areas separately.

Vertex conditions for quantum graphs and special classes of matrices with symmetries

Given a (formally symmetric) differential expression on a metric graph formed by a collection of intervals, to turn it into a self-adjoint operator one has to provide extra conditions connecting values of the functions and their derivatives at the vertices, so-called **vertex conditions**. Self-adjointness of the operator is required by quantum mechanics, also certain modifications like $\mathcal{PT}$-symmetric quantum mechanics are possible.. In many applications, the metric graph is given (determined by the physical system one aims to model), the differential operator is often taken equal to the Laplacian (influence of the potential on spectral properties of the system is secondary), hence it remains to choose vertex conditions. The most often used vertex conditions are standard vertex conditions requiring that the function is continuous at the vertices (condition welcomed by physicists) and the sum of oriented derivates sums up to zero at every vertex. These two conditions are often named by *continuity* and *Kirchhoff*'s conditions respectively. Standard condition appear naturally if the operator is defined by its (formal) quadratic form on continuous functions. So this condition is natural to require if nothing is known about the structure of the junction. All edges joined at such vertex are not only equal but the order they appear is irrelevant.

If one is interested in characterising all possible vertex conditions, then the (unitary) vertex scattering matrix $S_v(1)$ (denoted by U in the thesis) can serve as a perfect parameter: it not only determines vertex conditions in a unique way but has a clear physical interpretation (see formula (1.5) and Observation 1.4.2). Using this parametrisation it is not a hard task to characterise all vertex conditions possessing different symmetry properties (rotationally invariant, permutation invariant, *etc.*). From the physical point of view scattering matrices having equal transmission probabilities are of particular interest. The reason is two-fold:

- on one side only transmission probabilities (squared absolute values of the scattering coefficients forming the vertex scattering matrix) can be effectively measured in an experiment;
- on the other hand having no particular knowledge of how the edges are connected to each other leaves no other option than assuming that all transmission probabilities are equal.

It was noted by Y. Smilansky that standard vertex conditions corresponding to the scattering matrix

$$S_v = \begin{pmatrix} \frac{2}{n}-1 & \frac{2}{n} & \dots & \frac{2}{n} \\ \frac{2}{n} & \frac{2}{n}-1 & \dots & \frac{2}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{n} & \frac{2}{n} & \dots & \frac{2}{n}-1 \end{pmatrix},$$

where n is the degree of the vertex, also determining equal transmission probabilities, tend to -1 as $n \to \infty$ meaning that waves reflect completely from high degree vertices. Such behaviour is considered *unphysical* and naturally raises the question of charactering all equi-transmitting vertex scattering matrices called MPS-matrices (matrices with modular permutation symmetry) in the thesis..

The first part of the thesis is devoted to this problem and different aspects of equi-transmitting scattering matrices are discussed, in particular the following mathematical problems have been solved:

- characterisation of all vertex conditions leading to MPS scattering matrices;
- characterisation of all Hermitian MPS-matrices;
- construction of explicit examples of such matrices, including circulant matrices;
- connection to classical Hadamard matrices.

It is a pity that Dr. Turek ignored relevant publications where few of the results have already been proven, especially [2] based on [1] appeared earlier:

[1] M. Astudillo, PSEUDO-HERMITIAN LAPLACE OPERATORS ON STAR-GRAPHS: REAL SPECTRUM AND SELF-ADJOINTNESS, Master Thesis, Lunds Universitet, 2008.

[2] Maria Astudillo, Pavel Kurasov, and Muhammad Usman, *RT-symmetric Laplace operators on star graphs: real spectrum and self-adjointness*, Adv. Math. Phys., posted on 2015, Art. ID 649795, 9, DOI 10.1155/2015/649795. MR3442618

[3] P. Kurasov and B. Majidzadeh Garjani, *Quantum graphs: PT-symmetry and reflection symmetry of the spectrum*, J. Math. Phys. **58** (2017), no. 2, 023506, 14, DOI 10.1063/1.4975757. MR3608648

[4] Pavel Kurasov and Rao Ogik, *On equi-transmitting matrices*, Rep. Math. Phys. **78** (2016), no. 2, 199–218, DOI 10.1016/S0034-4877(16)30063-5. MR3569205

[5] Pavel Kurasov, Rao Ogik, and Amar Rauf, *On reflectionless equi-transmitting matrices*, Opuscula Math. **34** (2014), no. 3, 483–501, DOI 10.7494/OpMath.2014.34.3.483. MR3239078

[6] Rao, Wyclife Ogik: Quantum Graphs and Equi-transmitting Scattering Matrices, Lic. Thesis, Stockholm Univ., 2014.

[7] Amar Rauf: On Reflectionless Isoscattering Matrices, Master Thesis, Stockholm Univ. 2013.

[8] Miloslav Znojil, Quantum star-graph analogues of PT-symmetricsquare wells, Can. J. Phys. Vol. 90, 2012

The results proven by Dr. Turek and other authors complement each other in a nice way. Nevertheless analysis carried out by Dr. Turek is rather complete and demonstrates a high level of independence, especially since most of the papers on this subject were written independently of the former supervisor, Prof. P. Exner, who is one of the founders of the spectral theory of quantum graphs.

What I particularly like in these results is the ability to combine mathematical rigour with physical intuition. Let me just mention two examples:

- **quantum gates** with control constructed in Paper C:
  to construct the model one needs to understand extension theory for symmetric operators as well as physics of quantum filters;
- **generalised Hadamard and conference matrices** appeared in Paper F:
  here Dr. Turek demonstrates his ability to generalise classical circulant Hadamard and conference matrices from linear algebra following the needs of quantum graph models.

Periodic quantum graphs and Bethe-Sommerfeld conjecture

The second part of the thesis (Papers G–J) is devoted to periodic metric graphs with focus on Bethe-Sommerfeld property, *i.e.* whether the corresponding operator has a finite number of gaps in the spectrum or not. Another interesting problem is existence of flat bands – infinitely degenerated eigenvalues. Taking two- and three- dimensional lattices formed by metric graphs one may expect that the corresponding operator always has a finite number of gaps. It appears that quantum graphs on such lattices are not quite multidimensional and may have infinitely many gaps in the spectrum. This is the main result of Papers I and J.

Trying to understand whether the number of gaps is finite or infinite for lattices it is natural to start with few examples such as standard conditions, continuing with so-called scaling-invariant conditions leading to energy-independent vertex scattering matrices, and finally looking at most general vertex conditions. One may also instead of square lattices consider rectangular ones. The problem one faces is that not-standard vertex conditions lead to involved calculations making it hard to determine the number of gaps. So the Dr. Turek in collaboration with P. Exner after proving several *negative* results finally decided to analyse square lattice with delta conditions at the vertices. It appears that the result depend on number-theoretical properties of the ratio between the two periods. To derive the results it appeared necessary to introduce a certain generalisation of the Markov constant describing how good an irrational number could be approximated by rationals. Theorem 4.5 in Paper I states that properly choosing coupling parameter in the delta interaction one may achieve periodic quantum graphs with a finite number of gaps. To my opinion it is not only a beautiful result from spectral theory but also a result that nobody was expecting. It is based on a fine analysis of the interplay between the behaviour of open gaps and approximations of irrational numbers. It illustrates once more mathematical abilities of Dr. Turek to study a new area of mathematics that appears important for his studies and find out the only way out in the labyrinth of mathematical possibilities.

In the last Paper J Dr. Turek alone generalises these results to the case of cuboidal and even hyperrectangular lattices. It appears that precisely the same idea works: graphs with badly approximable ratios between the edge lengths may lead to periodic quantum graphs with a finite number of gaps, provided the delta couplings are properly chosen. One would expect such result after Paper I, but precise form of the involved quantities required careful analysis to be carried out.

Summary

Dr. Ondřej Turek is an active still young mathematical physicist with a broad network of collaborators and several interesting results obtained. There is no doubt that his competence is sufficient for Habilitation and current thesis just supports this conclusion. What is important is that he demonstrated ability to carry out independent research after his PhD thesis. Habilitation will help Dr. Turek in his further career and development.

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