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# Report on dr. Jiří Hrivnák's Habilitation Thesis MULTIVARIATE FOURIER–WEYL TRANSFORMS AND THEIR APPLICATIONS

The thesis consists of four chapters, introduction and conclusion as well as the list of references and included articles. The content of the thesis is based on the nine papers, published (except for one of them) in prominent physical journals "Journal of Mathematical Physicsór "Journal of Physics A: Mathematical and Theoretical". According to the subject of this thesis, let me divide the report in two parts: (i) mathematical results and (ii) applications.

(i) In general the mathematical part of the thesis concerns harmonic analysis related to the affine Weyl groups $W^{\text{aff}} \cong Q^\vee \rtimes W$ and the dual affine Weyl groups $\widehat{W}^{\text{aff}} \cong Q \rtimes W$ which are isomorphic to the semidirect products of the Weyl group $W$ with the root lattice $Q = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n$ and the dual root lattice $Q^\vee = \mathbb{Z}\alpha_1^\vee + \cdots + \mathbb{Z}\alpha_n^\vee$, respectively, where $\alpha_1, \ldots, \alpha_n$ are simple roots and $\alpha_1^\vee = 2\alpha_1/\langle\alpha_1, \alpha_1\rangle, \ldots, \alpha_n^\vee = 2\alpha_n/\langle\alpha_n, \alpha_n\rangle$ are dual simple roots of respective complex simple Lie algebra. The rescaled by $M \in \mathbb{N}$ dual affine Weyl group $\widehat{W}^{\text{aff}}_M = (MQ) \rtimes W$ is also considered.

As the kernels of Fourier–Weyl transforms (terminology taken from [A1–A4, A9]) are used

$$\phi^\sigma(a, b) \equiv \phi^\sigma_b(a) := \sum_{w \in W} \sigma(w)\, e^{2\pi i \langle wb, a\rangle}, \tag{1}$$

and

$$\zeta^\sigma(a, b) \equiv \zeta^\sigma_b(a) := \sum_{w \in W} \sigma(w) \operatorname{cas} 2\pi\langle wb,\, a\rangle, \tag{2}$$

where $a, b \in \mathbb{R}^n$, $\operatorname{cas} x := \cos x + \sin x$ and $\sigma : W \to \mathbb{Z}_2$ is a group homomorphism, chosen separately for the investigated subcase. The $W$–invariance of $\phi^\sigma$ and $\zeta^\sigma$ as well as the $2\pi i$–periodicity of the exponential function and the $2\pi$–periodicity of the cas function imply that the kernels (1) and (2), after restriction to the $W^{\text{aff}} \times \widehat{W}^{\text{aff}}$–invariant net

$$\frac{1}{M}(\varrho^\vee + P^\vee) \times (\varrho + P) \subset \mathbb{R}^n \times \mathbb{R}^n \tag{3}$$

and $W^{\mathrm{aff}} \times W^{\mathrm{aff}}$–invariant net

$$\frac{1}{M} P \times P \subset \mathbb{R}^n \times \mathbb{R}^n, \tag{4}$$

(for definitions of (3) and (4) see (1.5), (1.8) and (1.9) in the thesis) are determined by their values on the finite subnets

$$F_M^\sigma(\varrho, \varrho^\vee) \times \Lambda_M^\sigma(\varrho, \varrho^\vee) \subset \mathbb{R}^n \times \mathbb{R}^n, \tag{5}$$

and

$$F_{P,M}^\sigma \times \Lambda_{P,M}^\sigma \subset \mathbb{R}^n \times \mathbb{R}^n, \tag{6}$$

respectively. For definition of (5) and (6) see (2.8), (2.9) and (4.11), (4.12) in the thesis.

The above allows one to define the weighted scalar products

$$\langle f, g\rangle_{F_M^\sigma(\varrho,\varrho^\vee)} = \sum_{a\in F_M^\sigma(\varrho,\varrho^\vee)} \varepsilon(a) f(a)\overline{g(a)}, \tag{7}$$

and

$$\langle f, g\rangle_{F_{P,M}^\sigma} = \sum_{a\in F_{P,M}^\sigma} \varepsilon(a) f(a)\overline{g(a)}, \tag{8}$$

where $\varepsilon$ is a weight function, for its definition see (1.28), and shows that the following orthogonality relations for $\phi_b^\sigma$ are satisfied

$$\langle \phi_b^\sigma, \phi_{b'}^\sigma\rangle_{F_M^\sigma(\varrho,\varrho^\vee)} = c\,|W|\, M^n h_M^\vee(b)\, \delta_{b,b'}, \tag{9}$$

and

$$\langle \phi_b^\sigma, \phi_{b'}^\sigma\rangle_{F_{P,M}^\sigma} = d\,|W|\, M^n h_M(b)\, \delta_{b,b'}, \tag{10}$$

see (2.15) and (4.16). One of the personal contribution of dr. Jiří Hrivnák is finding the explicit forms of the sets $\Lambda_M^\sigma(\varrho,\varrho^\vee)$ and $\Lambda_{P,M}^\sigma$ and the normalization coefficients in the right-hand side of (9) and (10).

In consequence, from (9) and (10) follows the possibility to define for $f : F_M^\sigma(\varrho,\varrho^\vee) \to \mathbb{C}$ and $\widehat{f} : \Lambda_M^\sigma(\varrho,\varrho^\vee) \to \mathbb{C}$ as well as for $f : F_{P,M}^\sigma \to \mathbb{C}$ and $\widehat{f} : \Lambda_{P,M}^\sigma \to \mathbb{C}$ the forward and backward Fourier transforms:

$$\widehat{f}(b) = (c\,|W|\, M^n h_M^\vee(b))^{-1} \sum_{a\in F_M^\sigma(\varrho,\varrho^\vee)} \varepsilon(a) f(a) \overline{\phi_b^\sigma(a)}, \tag{11}$$

$$f(a) = \sum_{b\in\Lambda_M^\sigma(\varrho,\varrho^\vee)} \widehat{f}(b)\phi_b^\sigma(a). \tag{12}$$

and

$$\widehat{f}(b) = (d\,|W|\,M^n h_M(b))^{-1} \sum_{a\in F^{\sigma}_{P,M}} \varepsilon(a) f(a) \overline{\phi^{\sigma}_b(a)}, \tag{13}$$

$$f(a) = \sum_{b\in \Lambda^{\sigma}_{P,M}} \widehat{f}(b) \phi^{\sigma}_b(a). \tag{14}$$

These transforms are investigated in details in the included series of papers, where many important for applications examples are presented also. The most interesting results concerning the presented transforms are mentioned in the theorems and propositions listed in the introduction to the thesis.

In the papers [A5, A6] the case of the permutation group $W \cong S_n$ is investigated in detail. The most interesting parts of these papers concerns multivariable generalization of Chebyshev polynomials defined and investigated in [A6]. Various properties of these polynomials such as their recurrence relations and orthogonality (see Proposition 4.3 in [A6]) are shown. In [A7] relations between Weyl-orbit functions and modular data associated to WZNW conformal field theories are discussed . Also multiplication formulas for these functions are presented in [A7, A8].

(ii) In my opinion, one can distinguish the following applications of the mathematical results obtained in [A1–A9]:

(a) to problems in solid state physics and crystallography with Weyl group and even Weyl group symmetries, see the discrete Fourier-Weyl transforms and even transforms in [A1–A4];

(b) to the modeling of vibrations and transversal waves propagation of the graphene sheets, see type II extended Weyl orbit functions in [A9] and explicit solutions in subsection 4.4;

(c) to interpolation problems, e.g. by the construction of the numerical procedures for interpolation of the discretized model functions, see Propositions 3.1 and 3.2 in [A5], Examples 3.1 and 3.2 in [A6], Example 7.1 in [A9];

(d) for finding of explicit solution of some classes of the multivariate difference equations, see Example 4.1 in [A6];

(e) to generalization of Kac-Walton formulas and the Kac-Peterson matrices in WZNW conformal field theories, see relations (6)–(14) in [A7] and relation (50) in [A8].

Dr. Jiří Hrivnák has altogether published 25 articles in impacted journals. The publication databases Web of Science and Scopus contain 46 citations, without self-citations, to these impacted articles. There are two main directions in the candidate's research.

The first field of research encompasses contribution to the study of the structure of complex Lie algebras and their invariant characteristics. The set of invariant characteristics of Lie algebras is extended via generalized derivations and cocycles. The resulting invariants are applied to the classification of the graded contractions of $sl(3, \mathbb{C})$. The candidate's doctoral thesis is based on the publications from this field.

The second field of research, pertinent to the present habilitation thesis and consisting of 20 articles in impacted journals, is related to the Weyl group invariant functions and transforms. Summarizing, the content of the papers [A1–A9] written, except of the one case, by dr. Jiří Hrivnák with only one co-author, let me state that he is mature mathematical physicist and he contributed significantly to the discrete analysis methods in physics and signal processing theory. In my opinion his contribution to this domain is significant, therefore, **I recommend that his habilitation defense should proceed and that the "docent"title be awarded to dr. Jiří Hrivnák.**

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