OPPONENT'S REPORT

ON HABILITATION THESIS

"Algebraic Problems in Quantum Mechanics on Graphs"

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The subject and title of the thesis amalgamates the three concepts, viz., the formalism of Quantum Mechanics (QM), the specific research field of Quantum Mechanics on Graphs (QG) and, finally, a less sharply specified concept of Algebraic (methods or) Problems (AP).

The choice of such a title is clever because the connection between QM and QG is natural (see Appendices A - D) while the transition from QG to AP is necessary (in a way demonstrated in the other six Appendices). Marginally, let me add that the structure of the decuplet of the Appendices (representing the decuplet of the publications on which this thesis is based) is strongly reminiscent of the decuplet of elementary baryons. In this sense, in my eyes, Appendices A - D could have been assigned symbols Δ^{-} – Δ^{++} (i.e., the names of four zero-strangeness elementary particles), while the related triplet of the $S = -1$ baryons Σ^{*-}, Σ^{*0} and Σ^{*+} could mark the three QG-devoted (or, if you wish, Exner + Turek – coauthored) papers reprinted in Appendices

G, H and I. Next, in a sort of mathematical climax of the author's obvious personal preference and message, the doublet of the $S = -2$ particles Ξ^{*-} and Ξ^{*0} (containing two strange quarks) finds the unique one-to-one partnership in the two papers of Appendices E and F. Notice that they are both published in Linear Algebra Appl., the medium which is, bibliometrically speaking, the journal of the first-quartile scientific prestige and quality level. Finally, the last Appendix J (i.e., paper published in Rep. Math. Phys., exceptional in having O.T. as the only author) is, undoubtedly, as unique as the $S = -3$ particle Ω^- (composed of the three strange quarks)

I invested space and time to the latter introductory classification on purpose. Having read the thesis I detected a correspondence between such a decuplet-shaped classification of the subject of the thesis (i.e., between the size $|S|$ of the strangeness of the auxiliary elementary baryons) and the intensity of the author's fascination by the field (I will give more details below). Obviously, the temporarily earlier, $S = 0$ Appendices/publications still seem firmly rooted in QM and QG while, with time, the attention of O.T. definitely moved to the higher $|S|$, doomed to ultimately converge to the AP-related pure mathematics (which is not so surprising after all: see the O. T.'s affiliation).

For me (especially when I try to pretend that I am a physicist - or even nuclear physicist - see also my own affiliation), these developments are certainly able to evoke mixed feelings. Indeed, the best cited O. T.'s works are just the QM- and QG-related older papers, encountered here just in the "older" category of papers with, in my notation, vanishing strangeness, $|S| = 0$. Nevertheless, this only supports my preceding remark that the author put emphasis on his AP-related results and that he did not – correctly! – feel seduced by the wider impact of his physics-oriented papers among physicists (*pars pro toto* let me mention that, e.g., his QM- and QG-related paper [CET10a] which attracted as many as 42 WoS citations was not included in the main body of the thesis, being only mentioned, *in margo*, among the "other references" on p. 64.

Among the O.T.'s physics-related papers included in his introductory mainstream presentation (and carrying a clear trace of the co-authorship by T. Cheon) let me mention, as an illustrative sample, the text [TC12] published in European Physics Letters and reprinted in Appendix C. Via the study of threshold resonances the authors claimed that in a Fűlöp-Tsutsui-type star graph (with an external potential added on one of the lines) their analysis could lead, in principle at least, to a highly interesting design of the two quantum devices offering an adjustable quantum spectral filter, and/or a manipulative flow control of a new type.

Only after one moves through the wealth of the technical details emerging in the texts of the whole decuplet of the strongly or weakly QG-related analyses, one appreciates several, slowly emerging deeper remarks and AP-related discoveries. Firstly, the quantum-graph business (looking, at the first sight, as a deeply physics-related QM science) reveals its true nature as being – almost exclusively – mathematical and linear-algebraic. In the partly applied and partly abstract QG context let me mention, for illustration, the existence of a small comment on the relevance of the Hermitian unitary matrices in strictly QG paper [TC11] (on the phenomenological features of quantum graphs reflecting the scattering properties of vertices). Indeed, the idea got revitalized and upgraded, four years later, in a much more mathematical (i.e., on my classification scale, $|S| = 2$) analysis in papers [TC15] and [TG19].

Along these lines one starts understanding why O. T. did not mention some of its very well cited physics-oriented works (e.g., on the bent chain graphs) at all. He simply decided to concentrate on several true challenges related, e.g., to certain extremely interesting AP and number-theoretical aspects of the matrix and spectral theory. Thus, in paper [ET15], for example, one expects to find just an innocent-looking and more or less routine analysis of the properties of the spectral gaps. Nevertheless, to the reader's great surprise, the authors show that the existence of these gaps depends, critically, on the highly abstract number-theoretical properties of the ratios of the edge lengths.

In the closely related context of the study of the periodic QG spectra a significant part of the results of the thesis has been inspired by the very old Bethe-Sommerfeld conjecture. Some of its versions date back to the first half of the previous century (see, i.a., the difficult-to-find, i.e., not-quite-alphabetically ordered reference [SB33]). Extremely interesting connections have been studied and found in [ET17] and [Tu19]. In a way, even the physics-oriented readers of the thesis are finding a satisfaction when revealing the emergence of connections between the spectral aspects of the "highly phenomenological" cubic spatial structures of lattices, and the number-theoretical aspects of continued fractions (cf. [Tu19]), or between the role played by circulant matrices in the control of band spectra (and illustrated by multiple beautiful pictures of these spectra) in [ETT18]. Some of these results are truly unique: Personally I felt impressed by the theorems and examples of the explicit constructions of the quantum graphs with a prescribed number of the spectral gaps which can be infinite or finite, which can be non-zero or vanish, and the structure of which was explicit and constructive. Representing a forefront of the contemporary research: after all, many of the related (and, sometimes, technically rather complicated) questions still remain open at present.

A real pleasure and the definite reader's reward lies in the multiple mathematical details and new results, especially in the subfield of the old-fashioned though still incessantly fascinating theory of matrices. Even the sceptics (I know some) who would not quite believe in the ubiquitous presence of quantum graphs in the real world of quantum mechanics and/or in its applications in experimental setups will be forced to appreciate the deep mathematical appeal and beauty of the questions asked, in this context, on the (in the present opponent's classification, "non-zero strangeness") level by mathematicians and, besides spectral theorists, also by linear algebraists, and by number theorists. In particular, one of the most impressive and beautiful samples of the final renewal of meeting of mathematics with physics in the QG field may be seen in Table 4.1 (on p. 48). Here I am leaving this point without any explanatory comment, and I merely recommend everybody just to have a look.

In conclusion, the present referee feels pleased to inform the Scientific Council of CTU-FNSPE that the submitted habilitation thesis demonstrates, beyond any doubt, that its author, Ing. Ondřej Turek, PhD, is, in the field of Applied Mathematics, fully qualified for the corresponding promotion.

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